Stabilnost položaja ravnoteže lin. sistema dif. j-na sa konst. koef.

$x_1' = a_{11}x_1 + \dots + a_{1n}x_n$

$\vdots$

$x_n' = a_{n1}x_1 + \dots + a_{nn}x_n$

$a_{ij} \in \mathbb{R}$ (sistem ima jedinstv. rj.)

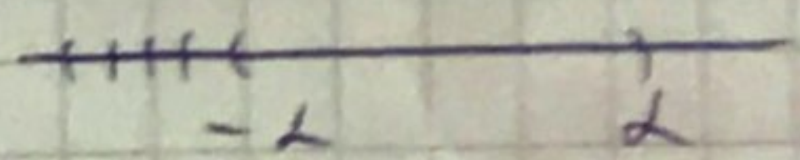
$X' = AX$ (1)

Šta je položaj ravnoteže? Kad su svi koef. $= 0$.

Pitamo se kad je stabilan ili nestabilan?

$\det(A - \lambda E) = 0$ – imamo n nula

$\lambda_j = \alpha_j + i\beta_j$, $j = 1, \dots, m$



Teorema: Ako postoji $\alpha > 0$ takvo da je $\alpha_j < -\alpha$, $\forall j = 1, \dots, m$, tada svako rješenje sistema (1) $x = \varphi(t)$, koje zadovoljava početni uslov $\varphi(t_0) = \varphi_0$ je stabilno.

- ako $\exists \alpha_j > 0$, tada su sva rješenja nestabilna.

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(ako je jedno nestab., svako je)

Teorema: Ako postoji $\alpha > 0$, t.d. $\alpha_j > \alpha$, tada je $x=0$ nestabilan položaj ravnoteže.

Teorema: Polinom $P(\lambda) = \det(A - \lambda E) = 0$ je stabilan akko je položaj ravnoteže $x=0$ asimptotski stabilan.

Lema: (Raus Hurvica)

Polinom $P(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \dots + a_n$ je stabilan akko su svi glavni minori matrice veći od nule

$$\begin{pmatrix} a_1 & 1 & 0 & \dots & & & 0 \\ a_3 & a_2 & a_1 & 1 & \dots & & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 \dots & 0 \\ & & & \ddots & & & \\ & & & & & & a_n \end{pmatrix}$$

$\Delta_1 = a_1 > 0$

$$\Delta_2 = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} > 0$$

$$\Delta_3 = \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0$$

1. Ispitati stabilnost:

$x_1' = ax_1 - x_2$

$x_2' = ax_2 - x_3$

$x_3' = ax_3 - x_1$

Teorema: Ako je $\alpha_j = 0$ samo za proste korijene, a u svim drugim slučajevima je $\alpha_j < 0$, tada je položaj $x=0$ stabilan

$$A = \begin{pmatrix} a & -1 & 0 \\ 0 & a & -1 \\ -1 & 0 & a \end{pmatrix}$$

$$\det(A-\lambda E)=0 \quad , \quad \begin{vmatrix} a-\lambda & -1 & 0 \\ 0 & a-\lambda & -1 \\ -1 & 0 & a-\lambda \end{vmatrix} = (a-\lambda)^3 - 1 = 0$$

$$(a-\lambda-1)((a-\lambda)^2 + a-\lambda+1) = 0$$

$$\lambda_1 = a-1 \; , \quad a^2 - 2\lambda a + \lambda^2 + a - \lambda + 1 = 0$$

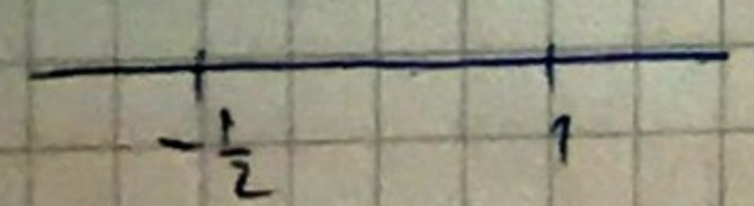
$$\lambda^2 - \lambda(2a+1) + a^2 + a + 1 = 0$$

$$\lambda_{1,2} = \frac{2a+1 \pm \sqrt{4a^2+4a+1-4a^2-4a-4}}{2}$$

$$\lambda_{1,2} = \frac{2a+1 \pm i\sqrt{3}}{2}$$

$$\Rightarrow \lambda_1 = a-1 \quad , \quad \lambda_{2,3} = a + \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$\Rightarrow$



$$ax_1 - x_2 = 0$$
$$ax_2 - x_3 = 0$$
$$ax_3 - x_1 = 0$$
$$\Rightarrow x_1 = x_2 = x_3 = 0$$

1° $a \geq 1 \Rightarrow \lambda_1 > 0$

$Re(\lambda_{2,3}) > 0 \Rightarrow$ položaj ravnoteže $x=0$ je nestabilan

2° $-\frac{1}{2} < a < 1 \Rightarrow \lambda_1 < 0$, $Re(\lambda_{2,3}) > 0$ $\Big\} \Rightarrow$ položaj ravnoteže je nestabilan

ne uključuje se jer bi bio stabilan

3° $a = -\frac{1}{2} \Rightarrow \lambda_1 = -\frac{3}{2}$, $Re(\lambda_{2,3}) = 0$ $\Big\}$ $x=0$ je stabilan

4° $a < -\frac{1}{2} \Rightarrow \lambda_1 < 0$, $Re(\lambda_{2,3}) < 0$ $\Big\} \Rightarrow$ $x=0$ je asimptotski stabilan položaj ravn.

2. Ispitati stabilnost rješenja dif. jednačine (primjenom leme R.H)

$x^{(5)} + 2x^{(4)} + 5x^{(3)} + 6x'' + 5x' + 2x = 0$

$P(\lambda) = \lambda^5 + 2\lambda^4 + 5\lambda^3 + 6\lambda^2 + 5\lambda + 2 = 0$

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 6 & 5 & 2 & 1 & 0 \\ 2 & 5 & 6 & 5 & 2 \\ 0 & 0 & 2 & 5 & 6 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$\Delta_1 = 2 > 0$

$\Delta_2 = 4 > 0$

$$\Delta_3 = \begin{vmatrix} 2 & 1 & 0 \\ 6 & 5 & 2 \\ 2 & 5 & 6 \end{vmatrix} = 8 > 0$$

$$\Delta_4 = \begin{vmatrix} 2 & 1 & 0 & 0 \\ 6 & 5 & 2 & 1 \\ 2 & 5 & 6 & 5 \\ 0 & 0 & 2 & 5 \end{vmatrix} = 7 > 0$$

$\Delta_5 = 14 > 0$

$\Rightarrow$ svi glavni minori su pozitivni, polinom je stabilan

$\Rightarrow$ rješenje je asimptotski stabilno.

Teoreme Ljapunova

$x' = F(x) \quad (1)$

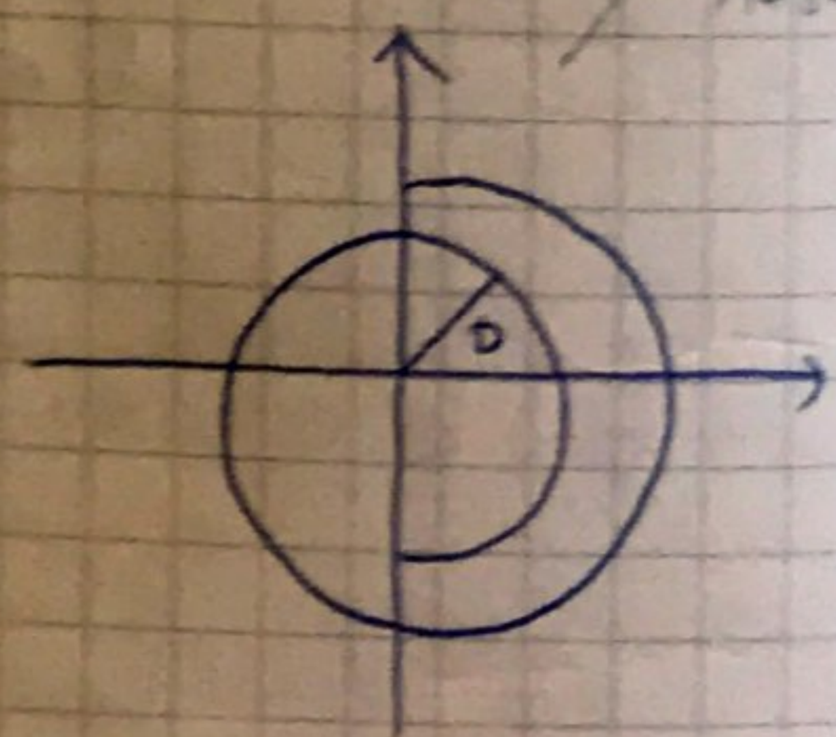
$x = 0$ - položaj ravnoteže

$x_1' = f_1(x) \quad , \quad x = x_1, \dots, x_n$

$\vdots$

$x_n' = f_n(x)$

Stabilno se smanjuje nestaje [illegible]



$D = \|x\| = \sqrt{x_1^2 + \dots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}$ Bitno je da je opadajuće

$$D' = \frac{1}{2} \frac{1}{\sqrt{\sum_{i=1}^{n} x_i^2}} \underbrace{\sum_{i=1}^{n} 2x_i \underbrace{x_i'}_{f_i(x)}}_{\leq 0 \Rightarrow D \downarrow \text{ i ima stabilan položaj ravn}}$$

u nuli je nula, u svakoj drugoj tački > 0

Uzmimo f-ju $V(x(t)) \in C^1(O(0))$ pozitivno definitna (negativno definitna)

$V(x(0)) = 0$

$V(x(t)) > 0$ - u cijeloj okolini tačke O
(< 0)

$V(x) = \alpha_1 x_1^2 + \ldots + \alpha_n x_n^2$, $\alpha_i \geq 0 \wedge \sum \alpha_i > 0$

ne smiju svi da budu 0 jer ne bi važilo ono gore

$\dot{V}(x) = 2\alpha_1 x_1 x_1' + \ldots + 2\alpha_n x_n x_n' =$

$= 2\alpha_1 x_1 f_1(x) + \ldots + 2\alpha_n x_n f_n(x) =$

$= (2\alpha_1 x_1, \ldots, 2\alpha_n x_n) \cdot (f_1(x), \ldots, f_n(x))$

$\dot{V}(x) = (\operatorname{grad} V(x), F(x)) \leq 0$ u $O(0)$.

Teorema: Za pozitivno definitnu f-ju V, t.d. $\forall x \in O(0)$ važi da je izvod duž sistema ≤ 0 $\Rightarrow$ tu f-ju V nazivamo funkcijom Ljapunova.

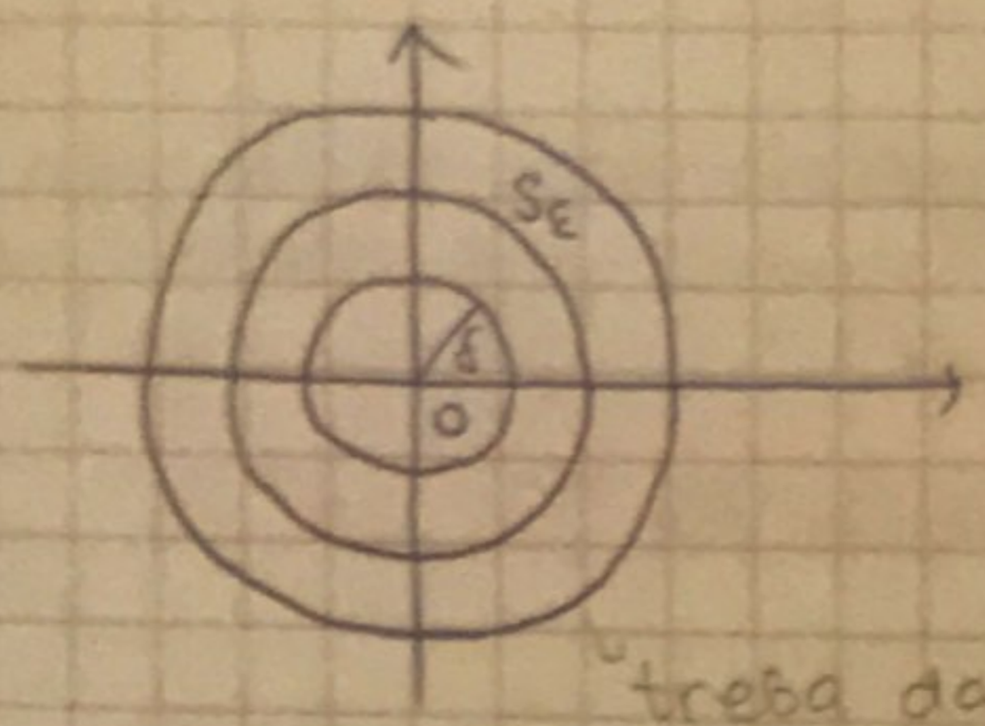
Teorema: (I teorema Ljapunova)

Ako u nekoj okolini tačke O postoji funkcija Ljapunova, tada je položaj ravnoteže $x = 0$ sistema (1) stabilan.

Dokaz: Imamo $O(0)$ t.d. u njoj postoji f-ja Ljapunova $\Rightarrow$ $\left.\begin{array}{l} V(x(t)) > 0, \ \forall x \in O(t) \\ \dot{V}(x(t)) \leq 0, \ \forall x\, O(0) \end{array}\right\} V\downarrow$

Izaberemo $\varepsilon > 0$ t.d. cijeli skup

$S_\varepsilon = \{x : \|x\| = \varepsilon\} \subset O(0)$.



treba da pokažemo da je položaj stabilan

Skup S_ε je zatvoren, f-ja opadajuća $\Rightarrow \exists \inf V(x(t)) = m$, tj. $\exists m : \min\limits_{S_\varepsilon} V(x,t) = m > 0$.

$V(0) = 0 \Rightarrow \exists \delta < \varepsilon$ t.d. $\|x\| = \delta$, $V(x(t)) < m$

Neka je $x = x(t)$ trajektorija, $x(t_0) = x_0$, i neka je $\|x\| < \delta$ (polazi iz male sfere). Treba da vidimo kako se ponaša ta trajektorija $\forall t \geq t_0$?

Pretpostavimo suprotno, $\exists T > t_0$ tako da: $\|x(t)\| < \varepsilon$, $t_0 < t < T$, a da je $\|x(T)\| = \varepsilon$.

Koliko je $V(x(T)) = ?$

$m < V(x,T) \leq V(x(t_0)) = V(x_0) < m$ ↯

pomičemo iz manjeg dok smo do većeg

$\Rightarrow$ položaj ravnoteže je stabilan.

Teorema: Ako u $O(0)$ postoji f-ja Lyapunova takva da $\forall x \in O(0)$ važi da je: $\mathring{V}(x) \leq -W(x)$, gdje je W - nepr. dif., pozitivno definitna f-ja u okolini tačke 0, tada je položaj ravnoteže $x = 0$ asimptotski stabilan.

Negacija: Ako u $O(0)$ postoji nepr. diferencijabilna funkcija $V(x)$, t.d. $V(0) = 0$, takva da je $V(x) \geq W(x)$, $\forall x \in O(0)$, tada je položaj ravnoteže $x = 0$ nestabilan.

1. Ispitati stabilnost položaja ravnoteže $x=0$ sistema:

$$x_1' = x_2 - x_1 + x_1 x_2$$
$$x_2' = x_1 - x_2 - x_1^2 - x_2^3$$

$V(x) = \alpha_1 x_1^2 + \alpha_2 x_1^2$

$\alpha_1 = \alpha_2 = 1 \Rightarrow V(x) = x_1^2 + x_2^2$

$\mathring{V}(x) = 2x_1 x_1' + 2x_2 x_2' = 2x_1(x_2 - x_1 + x_1 x_2) +$

$+ 2x_2(x_1 - x_2 - x_1^2 - x_2^3) = 2x_1x_2 - 2x_1^2 + 2x_1^2 x_2 + 2x_1 x_2 -$

$- 2x_2^2 - 2x_1^2 x_2 - 2x_2^4 = -2(x_1^2 - 2x_1x_2 + x_2^2 + x_2^4) =$

$= -2((x_1 - x_2)^2 + x_2^4) \leq 0, \quad \forall x_1, x_2 \in (0,0)$

$\Rightarrow$ položaj ravnoteže $x=0$ je stabilan

2. $x_1' = -x_1 - 2x_2 + x_1^2 x_2^2$

$x_2' = x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_1^3 x_2$

$V(x) = x_1^2 + x_2^2$

$\mathring{V}(x) = 2x_1x_1' + 2x_2x_2' = 2x_1(-x_1 - 2x_2 + x_1^2 x_2^2) +$

$+ 2x_2(x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_1^3 x_2) = -2x_1^2 - 2x_1x_2 - x_2^2 + x_1^3 x_2^2$

ne znamo kakvog je znaka, pa nam ne odg

Neka je $V(x) = \alpha_1 x_1^2 + \alpha_2 x_2^2$

$\mathring{V}(x) = 2\alpha_1 x_1 x_1' + 2\alpha_2 x_2 x_2' = -2\alpha_1 x_1^2 - 4\alpha_1 x_1 x_2 +$

$+ 2\alpha_1 x_1^3 x_2^2 + 2\alpha_2 x_1 x_2 - \alpha_2 x_2^2 - \alpha_2 x_1^3 x_2^2 =$

$= -2\alpha_1 x_1^2 - \alpha_2 x_2^2 + 2x_1x_2(\alpha_2 - 2\alpha_1) + x_1^3 x_2^2 (2\alpha_1 - \alpha_2)$

Neka je $\alpha_2 = 2\alpha_1$

$V(x) = x_1^2 + 2x_2^2$

$\Rightarrow \mathring{V}(x) = -2x_1^2 - 2x_2^2 = -2(x_1^2 + x_2^2) \Rightarrow x=0$ je asimpt. stabilan

3 $x_1' = x_1^3 - x_2$

$x_2' = x_1 + x_2^3$

$V(x_1, x_2) = \alpha_1 x_1^2 + \alpha_2 x_2^2$

$\dot{V}(x_1, x_2) = 2\alpha_1 x_1 x_1' + 2\alpha_2 x_2 x_2' = 2\alpha_1 x_1 (x_1^3 - x_2) +$

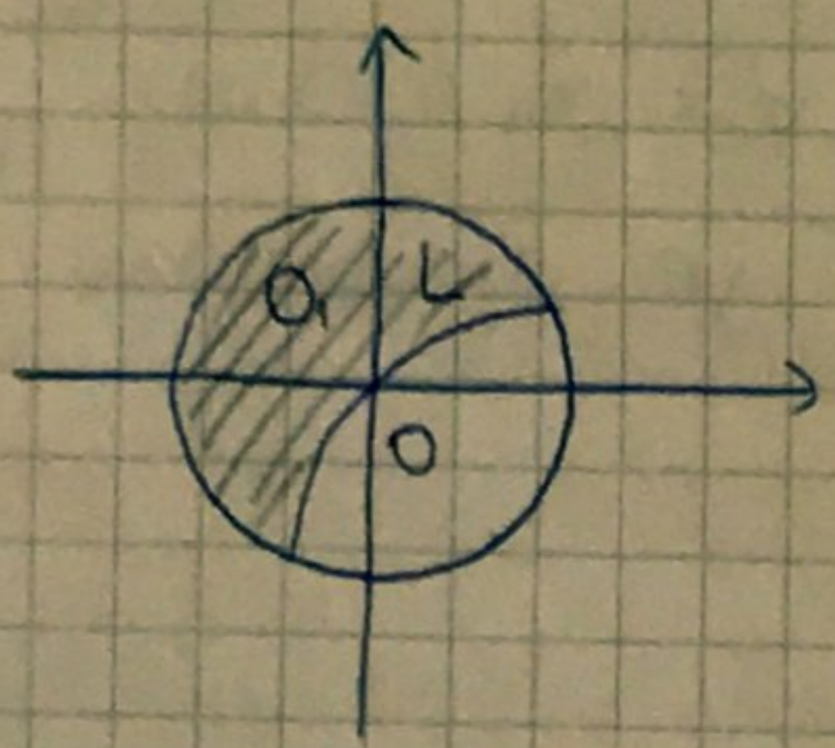
$+ 2\alpha_2 x_2 (x_1 + x_2^3) = 2(\alpha_1 x_1^4 - \alpha_1 x_1 x_2 + \alpha_2 x_1 x_2 + \alpha_2 x_2^4) =$

$= 2(\alpha_1 x_1^4 + \alpha_2 x_2^4 + x_1 x_2 (\alpha_2 - \alpha_1))$

$\alpha_1 = \alpha_2 = 1$

$\Rightarrow \dot{V}(x) = 2(x_1^4 + x_2^4) > 0 \Rightarrow$ nestabilan

$\sim$



$O_1 \subset O(0)$

$\{0\} \in L = \partial O_1$

Teorema:

Ako u O_1 postoji nepr. dif. funkcija V takva da:

1) $\forall x \in L$, $V(x) = 0$

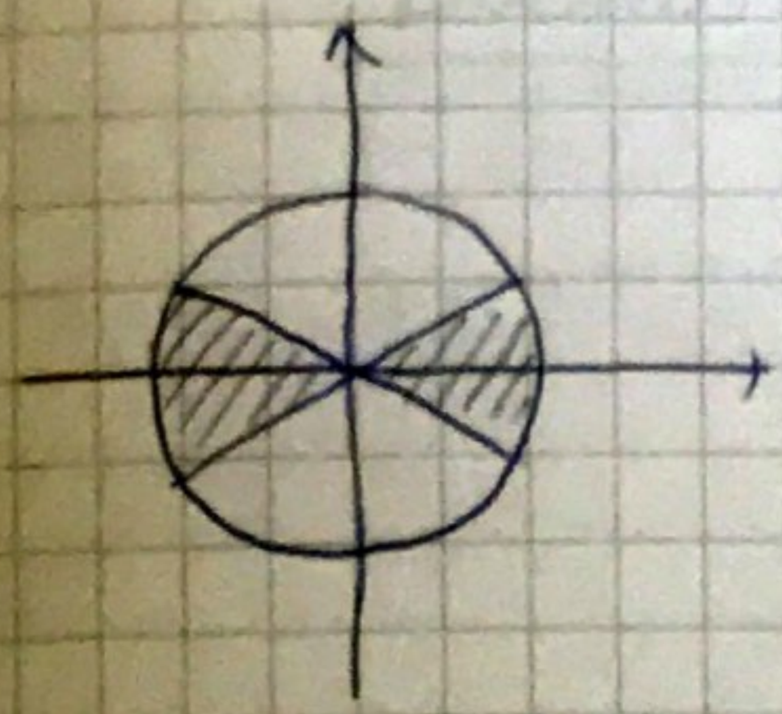
2) $\forall x \in O_1$, $V(x) > 0$

i $\dot{V}(x) = (\operatorname{grad} V(x), F(x)) > 0$

tada je položaj ravnoteže $x = 0$ nestabilan.

1. Ispitati stabilnost položaja ravnoteže $x = 0$ sistema: $x_1' = x_1^5 + x_2^3$

$x_2' = x_1^3 + x_2^5$



Neka je $O_1 = \{(x_1,x_2) : |x_1| > |x_2|\} \subset O(0)$

$L = \{(x_1,x_2) \mid |x_1| = |x_2|\} \ni (0,0)$

Ako je npr. $V(x_1,x_2) = x_1^2 - x_2^2$

$V(x_1,x_2) = 0, \quad \forall (x_1,x_2) \in L$

$V(x_1,x_2) > 0, \quad \forall (x_1,x_2) \in O_1$

$\dot{V}(x_1,x_2) = 2x_1x_1' - 2x_2x_2' = 2x_1^6 + 2x_1x_2^3 - 2x_1^3x_2 - 2x_2^6 =$

$= 2(x_1^6 - x_2^6 + x_1x_2(x_2^2 - x_1^2)) =$

$= 2((x_1^2 - x_2^2)(x_1^4 + x_1^2x_2^2 + x_2^4) + x_1x_2(x_2^2 - x_1^2))$

$\dot{V}(x_1,x_2) = 2(x_1^2 - x_2^2)(x_1^4 + x_1^2x_2^2 + x_2^4 - x_1x_2)$

- ne valja

$V(x_1,x_2) = x_1^4 - x_2^4$

$V(x_1,x_2) = 0, \quad \forall (x_1,x_2) \in L$

$V(x_1,x_2) > 0, \quad \forall (x_1,x_2) \in O$

$\dot{V}(x_1,x_2) = 4x_1^3x_1' - 4x_2^3x_2' =$

$= 4(x_1^3(x_1^5 + x_2^3) - x_2^3(x_1^3 + x_2^5)) = 4(x_1^8 - x_2^8) > 0$, $\forall (x_1,x_2) \in O_1$ ($|x_1| > |x_2|$)

$\Rightarrow$ $x = 0$ nestabilan